

Quantum Gravitational Collapse and Hawking Radiation in 2+1 Dimensions

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ABSTRACT

We develop the canonical theory of gravitational collapse in 2+1 dimensions with a negative cosmological constant and obtain exact solutions of the Wheeler–DeWitt equation regularized on a lattice. We employ these solutions to derive the Hawking radiation from black holes formed in all models of dust collapse. We obtain an (approximate) Planck spectrum near the horizon characterized by the Hawking temperature $T_H = \hbar\sqrt{G\Lambda M}/2\pi$, where M is the mass of a black hole that is presumed to form at the center of the collapsing matter cloud and $-\Lambda$ is the cosmological constant. Our solutions to the Wheeler-DeWitt equation are exact, so we are able to reliably compute the greybody factors that result from going beyond the near horizon region.

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I. INTRODUCTION

A principal goal of modern theoretical physics is the construction of a consistent quantum theory of gravity. In fact much effort has been directed at this problem over several decades with the result that there are at present several proposals on the table. However, it is probably fair to say that none of the proposals currently under development is either fully understood, free of ambiguities or universally accepted [1].

In the absence of either a full quantum theory of gravity or of any direct experimental input, it seems worthwhile to address the quantization of particular models by the application of as wide a variety of techniques as possible. Among the most interesting and best understood models are those with spherical symmetry, while an especially interesting model with

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spherical symmetry is that of self-gravitating matter. One model of gravitational collapse that is very well understood on the classical level is that of Lemaître [2], Tolman [3], and Bondi [4] (LTB), which describes a self-gravitating cloud of pressureless dust. The system is known to develop both covered and naked singularities and therefore a successful quantization of these models has the potential to address also the fate of the naked singularity.

Ideally, one would like to have a quantum model that is able to predict what the final state of collapse will be, in particular whether or not and under what conditions the formation of a singularity may be avoided [1, 5]. The model should also explain some features of the end-states. For example, it has been known for a very long time from semi-classical arguments that black holes behave as heat reservoirs and evaporate as black bodies with a characteristic temperature that depends only upon their conserved charges [6, 7, 8]. Explaining these thermodynamic properties in terms of their microstates would be a desirable feature of any quantum theory of collapse. Further, semi-classical arguments indicate that the nature of particle creation from a naked singularity is significantly different from a black hole and possibly even that they are unstable [9]; however, because the semi-classical treatment breaks down when curvatures reach the Planck scale, a full quantum treatment is essential to understanding their evolution. This is another area upon which a good quantum model of collapse should shed some light.

In order to examine these problems we have set up a canonical quantization program [10, 11] to describe a self-gravitating dust cloud using the variables introduced in [12]. In our work, the classical geometrodynamical constraints of the gravity-dust system were given in terms of a canonical chart consisting of the mass contained within spherical shells, the area radius, the dust proper time, and their conjugate momenta. The diffeomorphism constraint was used to eliminate the momentum conjugate to the mass function and this procedure was shown to result in a much simpler constraint that was able to take the place of the original Hamiltonian constraint. Dirac's quantization then led to the Wheeler–DeWitt equation for the wave functional describing the quantum collapse. Later it was shown that the WKB treatment of the Schwarzschild black hole in this canonical picture describes Hawking radiation [13]. To go beyond the WKB approximation, the Wheeler–DeWitt equation was then regularized on a lattice and quantum corrections to the radiation were proposed in [14]. However, the lattice regularization turned out not to be diffeomorphism invariant. This issue was addressed in [15]. When care is taken to ensure that the momentum constraint is fulfilled in the continuum limit, the lattice wave functional becomes described by the Hamilton–Jacobi equation and two additional constraints which uniquely determine the factor ordering and make it possible to obtain exact solutions for the wave functional. Subsequently, we reexamined the Hawking radiation for the general case of non-marginal collapse [16], recovering the Planck spectrum near the horizon and additional grey-body factors as the near-horizon approximation was relaxed.

In this paper we turn our attention to the situation in which a negative cosmological constant is present. We feel it is desirable to first adapt our program to the lower dimensional 2+1-dimensional case, where dust collapse has been examined in detail on the classical [17, 18] and the semi-classical level [19] by two of us. Moreover, the BTZ black hole [20, 21], which is the unique classical end state of the collapse, is reasonably well understood on

the quantum level from the point of view of the AdS/CFT correspondence [22, 23]. Thus it may be possible in the future to compare the degrees of freedom employed by the two approaches. Most of the calculations are in close analogy to the calculations presented in our earlier papers on the LTB model, but there are various subtle differences (boundary conditions, relative factors in expressions) that need to be addressed; therefore, we present the analysis in sufficient detail to provide a self-contained presentation.

Our paper is organized as follows. In Sec. II we review the classical model of self-gravitating dust in 2+1 dimensions and set up the canonical formalism in ADM variables. In Sec. III we transform to the more transparent Kuchař-type variables appropriate to the presence of a negative cosmological constant. Here we determine the mass function in terms of the ADM variables and go on to express the canonical constraints in a chart consisting of the radius, R , the mass function, F , the dust proper time, τ and their conjugate momenta. We also take special care to address the boundary action. In Sec. IV we obtain the Wheeler–DeWitt equation and present the exact quantum states appropriate to a lattice regularization. In Sec. V we show how Hawking radiation, together with the appropriate grey-body factors, can be recovered from our exact solutions. We summarize our results in Sec. VI and conclude with a few comments on potential future developments.

II. THE MODEL

A. The classical solutions

We are concerned with a self-gravitating, pressureless dust cloud, described by the energy-momentum tensor $T^\mu{}_\nu = \varepsilon U^\mu U_\nu$, in 2+1 dimensions with a negative cosmological constant $-\Lambda$, $\Lambda > 0$. The metric may be given in comoving, synchronous coordinates as

$$ds^2 = -d\tau^2 + e^{2b(\tau,\rho)} d\rho^2 + R(\tau,\rho)^2 d\varphi^2, \quad (1)$$

where τ is the dust proper time, and ρ labels dust shells of physical (curvature) radius $R(\tau,\rho)$. Inserting this line element into Einstein’s equations leads to the relation [18]

$$e^{2b(\tau,\rho)} = \frac{(\partial_\rho R)^2}{2(E - F)}, \quad (2)$$

where $E(\rho)$ and $F(\rho)$ are time independent but otherwise arbitrary functions of ρ , which satisfy

$$2\pi G\varepsilon(\tau,\rho) = \frac{\partial_\rho F}{R(\partial_\rho R)}, \quad (\partial_\tau R)^2 = 2E - \Lambda R^2. \quad (3)$$

We note that the gravitational constant G has the physical dimension of an inverse mass in 2+1 dimensions (we set $c = 1$ throughout); the quantities F and E are dimensionless.

The case of collapse is described by the condition $\partial_\tau R < 0$. There is still a freedom to rescale the shell label ρ : this can be fixed by demanding that

$$R(0,\rho) = \rho, \quad (4)$$

so that at the initial epoch ($\tau = 0$) the label coordinate is equal to the curvature radius R . This allows us to express the functions $E(\rho)$ and $F(\rho)$ in terms of the energy density at $\tau = 0$. From (3) we find

$$\begin{aligned} F(\rho) &= 2\pi G \int_0^\rho \rho' \varepsilon(0, \rho') d\rho' \\ E(\rho) &= [\partial_\tau R(0, \rho)]^2 + \Lambda \rho^2 \end{aligned} \quad (5)$$

The physical interpretation of these relations is that $2F(\rho)$ represents the gravitating mass inside the shell labeled by ρ , and $E(\rho)/2$ is the total energy per unit mass of that shell. Therefore $F(\rho)$ is generally known as the “mass function” of the shell and $E(\rho)$ as its “energy function”.

The solution to (3),

$$R(\tau, \rho) = \sqrt{\frac{2E}{\Lambda}} \sin \left(-\sqrt{\Lambda} \tau + \sin^{-1} \sqrt{\frac{\Lambda}{2E}} \rho \right), \quad (6)$$

shows that the shell labeled by ρ reaches a curvature radius $R = 0$ at the time

$$\tau_0(\rho) = \frac{1}{\sqrt{\Lambda}} \sin^{-1} \left(\sqrt{\frac{\Lambda}{2E}} \rho \right). \quad (7)$$

At $\tau = \tau_0(\rho)$ the shell becomes singular; therefore, τ can take values no larger than $\tau_0(\rho)$. A detailed analysis, performed in [18], shows that only those shells that obey the relation $2F > \Lambda R^2$ become trapped. Not only does this mean that, for a black hole to form, $F > 0$ but also that a shell becomes trapped when it collapses to a size less than $\sqrt{2F/\Lambda}$.

The solutions can be matched to an exterior BTZ black hole [17, 18],

$$ds^2 = -(\Lambda R^2 - GM)dT^2 + \frac{dR^2}{\Lambda R^2 - GM} + R^2 d\varphi^2, \quad (8)$$

at some boundary that is specified by a fixed shell (the outermost shell), labeled ρ_b . The matching requires the mass function at the boundary to be related to the mass parameter of the BTZ black hole according to $F(\rho_b) = GM/2$.

B. The Hamiltonian

The ADM metric with circular symmetry for a 2+1 dimensional system takes the form

$$ds^2 = -N(t, r)dt^2 + L^2(t, r)(dt + N^r dr)^2 + R^2(t, r)d\varphi^2, \quad (9)$$

where N is the lapse function and N^r is the only shift that survives the symmetry. In terms of these metric components, the Einstein-Hilbert action can be written as

$$S_{\text{EH}} = G^{-1} \int dt \int_0^\infty dr \left[\frac{N}{L^2} (L'R' - LR'' + \Lambda L^3 R) - \frac{1}{N} [(N^r L)' - \dot{L}][N^r R' - \dot{R}] \right], \quad (10)$$

after integrating out the angular dependence. From here, we obtain the momenta conjugate to $R(t, r)$ and $L(t, r)$,

$$\begin{aligned} P_L &= -\frac{1}{GN}[\dot{R} - N^r R'], \\ P_R &= -\frac{1}{GN}[\dot{L} - (N^r L)'], \end{aligned} \quad (11)$$

and the action can be put in the form

$$S_{\text{EH}} = \int dt \int_0^\infty dr [P_L \dot{L} + P_R \dot{R} - N\mathcal{H}^g - N^r \mathcal{H}_r] + S_{\partial\Sigma}, \quad (12)$$

where $S_{\partial\Sigma}$ is a boundary term to be discussed below. Because no time derivatives of the lapse or shift occur in the action, they are Lagrange multipliers. The Hamiltonian and momentum (diffeomorphism) constraints are given respectively by

$$\begin{aligned} \mathcal{H}^g &= -GP_L P_R - G^{-1} \Lambda R L + G^{-1} \left(\frac{R'}{L} \right)' \approx 0 \\ \mathcal{H}_r &= R' P_R - L P_L' \approx 0. \end{aligned} \quad (13)$$

The total action is the sum of (12) and an action S_{D} that describes the dust. The canonical formalism for dust was developed in [24] and elaborated in [25]. We consider only non-rotating dust, for which

$$S_{\text{D}} = \int dt \int_0^\infty dr [P_\tau \dot{\tau} - N\mathcal{H}^d - N^r \mathcal{H}_r^d], \quad (14)$$

where the Hamiltonian and momentum constraints are given by

$$\begin{aligned} \mathcal{H}^d &= P_\tau \sqrt{1 + \frac{\tau'^2}{L^2}}, \\ \mathcal{H}_r^d &= \tau' P_\tau \end{aligned} \quad (15)$$

The matter configuration will also act as a time-keeper for the quantum theory. In principle one could think to use a fundamental field for this purpose, but this would make the problem much less tractable and, as we will see, the main features of the theory are already contained in the dust model.

It is easy to verify that the Poisson-bracket algebra of the constraints closes and the system is first class. With the total Hamiltonian denoted by \mathbb{H} , the Hamiltonian equations of motion read

$$\begin{aligned} \dot{R} &= \{R, \mathbb{H}\}_{\text{PB}} = \frac{\delta \mathbb{H}}{\delta P_R} = -NGP_L + N^r R', \\ \dot{L} &= \{L, \mathbb{H}\}_{\text{PB}} = \frac{\delta \mathbb{H}}{\delta P_L} = -NGP_R + (N^r L)', \end{aligned}$$

$$\begin{aligned}
\dot{\tau} &= \{\tau, \mathbb{H}\}_{\text{PB}} = \frac{\delta \mathbb{H}}{\delta P_\tau} = N \sqrt{1 + \frac{\tau'^2}{L^2}} + N^r \tau', \\
\dot{P}_R &= \{P_R, \mathbb{H}\}_{\text{PB}} = -\frac{\delta \mathbb{H}}{\delta R} = -\frac{N''}{GL} + \frac{N' L'}{GL^2} + G^{-1} N \Lambda L + (N^r P_R)', \\
\dot{P}_L &= \{P_L, \mathbb{H}\}_{\text{PB}} = -\frac{\delta \mathbb{H}}{\delta L} = -\frac{N' R'}{GL^2} + G^{-1} N \Lambda R + \frac{N \tau'^2 P_\tau}{L^2 \sqrt{L^2 + \tau'^2}} + N^r P_L', \\
\dot{P}_\tau &= \{P_\tau, \mathbb{H}\}_{\text{PB}} = -\frac{\delta \mathbb{H}}{\delta \tau} = \left(\frac{N \tau' P_\tau}{L \sqrt{L^2 + \tau'^2}} + N^r P_\tau \right)', \tag{16}
\end{aligned}$$

and from them the equations in (3) can be recovered in the gauge $\tau = t$.

C. The fall-off conditions

We consider only mass functions which are such that at infinity the solutions approach the BTZ spacetime. This would be true in models in which the collapsing metric either asymptotically approaches or is smoothly matched to an exterior BTZ black hole at some boundary ρ_b . It is then necessary to adopt the following fall-off conditions at spatial infinity:

$$\begin{aligned}
R(t, r) &\rightarrow r + \mathcal{O}^\infty(r^{-2}) \\
L(t, r) &\rightarrow \frac{r^{-1}}{\Lambda^{1/2}} + \frac{GM_+(t)r^{-3}}{\Lambda^{3/2}} + \mathcal{O}^\infty(r^{-4}) \\
P_R(t, r) &\rightarrow \mathcal{O}^\infty(r^{-4}) \\
P_L(t, r) &\rightarrow \mathcal{O}^\infty(r^{-2}) \\
N(t, r) &\rightarrow \left[\sqrt{\Lambda} r - \frac{GM_+(t)r^{-1}}{\sqrt{\Lambda}} + \mathcal{O}^\infty(r^{-3}) \right] N_+(t) + \mathcal{O}^\infty(r^{-4}) \\
N^r(t, r) &\rightarrow \mathcal{O}^\infty(r^{-2}) \tag{17}
\end{aligned}$$

On the other hand, the fall-off conditions at the origin depend sensitively on the conditions we place on the energy and mass functions. If we wish to avoid shell crossing singularities then the energy function must be positive, with negative slope. Near the center ($r = 0$) we take $E = \sum_n E_n r^n$, with $E_0 > 0$ and $E_1 < 0$. Likewise, take $F = \sum_n F_n r^n$ with $F_0 > 0$. In this way, we determine the following conditions as $r \rightarrow 0$:

$$\begin{aligned}
R(t, r) &\rightarrow a(t) + b(t)r + \mathcal{O}^0(r^3), \\
L(t, r) &\rightarrow \gamma b(t) + \mathcal{O}^0(r), \\
P_R(t, r) &\rightarrow P_{R0}(t) + \mathcal{O}^0(r), \\
P_L(t, r) &\rightarrow P_{L0}(t) + \mathcal{O}^0(r),
\end{aligned}$$

$$N(t, r) \rightarrow \gamma N_0(t) + \mathcal{O}^0(r^2),$$

$$N^r(t, r) \rightarrow \mathcal{O}^0(r). \quad (18)$$

They are consistent with the constraints (13) and are preserved by the equations of motion. With these fall-off conditions, the only non-vanishing boundary variations that arise read

$$\int dt N_+(t) M_+(t), \quad (19)$$

where $N_+(t)$ is the lapse function as $r \rightarrow \infty$ and $GM_+ = 2F(r \rightarrow \infty)$ is the ADM mass and a similar contribution at $r = 0$,

$$- \int dt N_0(t) M_0(t) \quad (20)$$

where $N_0(t)$ is the lapse function as $r \rightarrow 0$ and $GM_0 = 2F(0)$. To avoid the conclusion that N_+ and N_0 freeze the evolution at the respective boundaries, the boundary terms must be cancelled by an appropriate boundary action. This can be achieved by adding the surface action

$$S_{\partial\Sigma} = - \int dt N_+(t) M_+(t) + \int dt N_0(t) M_0(t) \quad (21)$$

In the next section we show how this surface action can be absorbed into the hypersurface action by introducing a new canonical chart.

III. CANONICAL TRANSFORMATIONS

A. Mass function in terms of the canonical variables

Now consider an embedding of the ADM metric (9) in the spacetime described by the metric (1),

$$ds^2 = -d\tau^2 + \frac{\tilde{R}^2}{2(E-F)} d\rho^2 + R^2 d\varphi^2, \quad (22)$$

where $R = R(\tau, \rho)$, $\tilde{R} = \partial_\rho R$, and we will use $R^* = \partial_\tau R$ as opposed to a prime and a dot for derivatives with respect to the ADM labels r and t , respectively. We shall set $G = 1$ in the following to simplify the expressions. Let $\bar{R} = \tilde{R}/\sqrt{2(E-F)}$, then

$$\begin{aligned} L^2 &= \bar{R}^2 \rho'^2 - \tau'^2, \\ N &= \frac{\bar{R}}{L} (\dot{\tau} \rho' - \dot{\rho} \tau'), \\ N^r &= \frac{\bar{R}^2 \dot{\rho} \rho' - \dot{\tau} \tau'}{L^2}. \end{aligned} \quad (23)$$

Inserting these into the momenta (11), we find that

$$LP_L = -\frac{1}{\bar{R}(\dot{\tau} \rho' - \dot{\rho} \tau')} \left[\dot{R}(\bar{R}^2 \rho'^2 - \tau'^2) - (\bar{R}^2 \dot{\rho} \rho' - \dot{\tau} \tau') R' \right]. \quad (24)$$

Derivatives with respect to the ADM time can be exchanged for derivatives with respect to the proper time using

$$\begin{aligned}\dot{R} &= R^*\dot{\tau} + \tilde{R}\dot{\rho} = R^*\dot{\tau} + \overline{R}\sqrt{2(E-F)}\dot{\rho}, \\ R' &= R^*\tau' + \tilde{R}\rho' = R^*\tau' + \overline{R}\sqrt{2(E-F)}\rho',\end{aligned}\tag{25}$$

which gives

$$LP_L = -\frac{R'R^*}{\sqrt{2(E-F)}} + \frac{(R^{*2} - 2(E-F))}{\sqrt{2(E-F)}}\tau'.\tag{26}$$

Substituting Einstein's equation (3), $R^{*2} = 2E - \Lambda R^2$, then yields the simplified form

$$LP_L = \mp \frac{R'\sqrt{2E - \Lambda R^2}}{\sqrt{2(E-F)}} - \frac{\Lambda R^2 - 2F}{\sqrt{2(E-F)}}\tau',\tag{27}$$

which may be solved for τ' :

$$\tau' = -\frac{LP_L\sqrt{2(E-F)}}{\Lambda R^2 - 2F} \mp \frac{R'\sqrt{2E - \Lambda R^2}}{\Lambda R^2 - 2F}.\tag{28}$$

Substituting this into the expression for L in (23),

$$L^2 = \overline{R}^2\rho'^2 - \tau'^2,\tag{29}$$

and solving for F gives an expression for the mass function in terms of the canonical variables,

$$F = \frac{1}{2} \left[P_L^2 - \frac{R'^2}{L^2} + \Lambda R^2 \right].\tag{30}$$

For future reference we introduce the function \mathcal{F} defined by

$$\mathcal{F} = \Lambda R^2 - 2F = \frac{R'^2}{L^2} - P_L^2.\tag{31}$$

$\mathcal{F} = 0$ determines the apparent horizon and \mathcal{F} will play an important role in the quantum theory. One can check that though \mathcal{F} appears in the denominator of (28), τ' continues to be well behaved across the horizon, as expected. (We note the difference in the definition of \mathcal{F} compared to the LTB case where one has $\mathcal{F} = 1 - F/R$ [15].)

B. New variables

As in the case of the Schwarzschild black hole [12] and the LTB collapse model in 3+1 dimensions [10], one can make a canonical transformation that elevates the mass function to a canonical variable. Interestingly, the expressions are similar to those presented by us in earlier papers on LTB collapse [15, 16]. By directly taking Poisson brackets the momentum conjugate to the mass function is found to be

$$P_F = \frac{LP_L}{\Lambda R^2 - 2F} = \frac{LP_L}{\mathcal{F}}.\tag{32}$$

Looking for a canonical transformation that would take the diffeomorphism constraint to

$$\mathcal{H}_r = R'P_R - LP_L' \rightarrow R'\overline{P}_R + F'P_F, \quad (33)$$

we find a simple expression for \overline{P}_R :

$$\overline{P}_R = P_R + \frac{\Lambda RLP_L}{\mathcal{F}} - \frac{\Delta}{L^2\mathcal{F}}, \quad (34)$$

where $\Delta = (LP_L)'R' - (LP_L)R''$. Our momentum constraint becomes then indeed

$$\mathcal{H}_r = R'\overline{P}_R + F'P_F \approx 0, \quad (35)$$

but we must first show that the transformation from the old set $\{R, L, P_R, P_L\}$ to the set $\{R, F, \overline{P}_R, P_F\}$ is canonical. We can do this by explicitly constructing the generator of the transformation. Denote it by \mathbb{F} . Since we already know two coordinates (R and F) and one conjugate momentum (P_F), we use

$$\begin{aligned} P_L(r) &= \int dr' P_F(r') \frac{\partial F(r')}{\partial L(r)} + \frac{\delta \mathbb{F}}{\delta L(r)}, \\ P_R(r) &= \overline{P}_R(r) + \int dr' P_F(r') \frac{\partial F(r')}{\partial R(r)} + \frac{\delta \mathbb{F}}{\delta R(r)}, \\ 0 &= \int dr' P_F(r') \frac{\partial F(r')}{\partial P_L(r)} + \frac{\delta \mathbb{F}}{\delta P_L(r)}, \\ 0 &= \int dr' P_F(r') \frac{\partial F(r')}{\partial P_R(r)} + \frac{\delta \mathbb{F}}{\delta P_R(r)}. \end{aligned} \quad (36)$$

The last equation in (36) tells us that $\mathbb{F} = \mathbb{F}[R, L, P_L]$. The third equation gives

$$P_F(P_L) + \frac{\delta \mathbb{F}}{\delta P_L} = 0 \Rightarrow \frac{\delta \mathbb{F}}{\delta P_L} = \frac{LP_L^2}{P_L^2 - R'^2/L^2}, \quad (37)$$

and therefore

$$\mathbb{F} = \int dr \left[LP_L - R' \tanh^{-1} \frac{LP_L}{R'} \right] + \mathbb{F}_1[L, R], \quad (38)$$

whereas, from the first equation,

$$P_L + \frac{R'^2 P_L}{L^2 P_L^2 - R'^2} = \frac{\delta \mathbb{F}}{\delta L} = P_L + \frac{R'^2 P_L}{L^2 P_L^2 - R'^2} + \frac{\delta \mathbb{F}_1}{\delta L}, \quad (39)$$

showing that $\mathbb{F}_1 = \mathbb{F}_1[R]$. Take \mathbb{F}_1 to be independent of R , a constant, and let us calculate \overline{P}_R from the resulting \mathbb{F} :

$$\overline{P}_R = P_R - \int dr' P_F(r') \frac{\partial F(r')}{\partial R(r)} - \frac{\delta \mathbb{F}}{\delta R} \quad (40)$$

Integrating by parts,

$$\overline{P}_R = P_R - \left(\frac{P_F R'}{L^2} \right)' + P_F \Lambda R - \frac{\delta \mathbb{F}}{\delta R}$$

$$\begin{aligned}
&= P_R + \frac{\Lambda R L P_L}{\mathcal{F}} - \frac{(L P_L / R')'}{1 - (L P_L / R')^2} \\
&= P_R + \frac{\Lambda R L P_L}{\mathcal{F}} - \frac{\Delta}{L^2 \mathcal{F}},
\end{aligned} \tag{41}$$

we find precisely the new candidate momentum in (34). The transformation from $\{R, L, P_R, P_L\}$ to $\{R, F, \bar{P}_R, P_F\}$ is generated by \mathbb{F} .

We now want to write the Hamiltonian in terms of the new variables, $\{R, \bar{P}_R, F, P_F\}$. To do so, we use the two equations

$$\begin{aligned}
P_L^2 - \frac{R'^2}{L^2} &= -\mathcal{F}, \\
L P_L &= \mathcal{F} P_F.
\end{aligned} \tag{42}$$

Inserting the second into the first gives

$$\frac{P_F^2 \mathcal{F}^2 - R'^2}{L^2} = -\mathcal{F} \Rightarrow L^2 = \frac{R'^2 - P_F^2 \mathcal{F}^2}{\mathcal{F}} \tag{43}$$

and therefore

$$P_L^2 = \frac{P_F^2 \mathcal{F}^3}{R'^2 - P_F^2 \mathcal{F}^2} \tag{44}$$

as well as

$$P_R = \bar{P}_R + \Lambda R P_F - \frac{\Delta}{P_F^2 \mathcal{F}^2 - R'^2}. \tag{45}$$

Putting it all together, we find

$$\mathcal{H}^g = -\frac{1}{L} [\mathcal{F} P_F \bar{P}_R + \mathcal{F}^{-1} R' F'] \tag{46}$$

which is of the same form as the expressions for the Schwarzschild black hole and LTB collapse in 3+1 dimensions, cf. Equation (42) in [15]. The action in the new canonical variables then reads

$$S_{\text{EH}} = \int dt \int_0^\infty dr \left(P_\tau \dot{\tau} + \bar{P}_R \dot{R} + P_F \dot{F} - N \mathcal{H}^g - N^r \mathcal{H}_r \right) + S_{\partial\Sigma} \tag{47}$$

with the new constraints (46) and

$$\mathcal{H}_r = R' \bar{P}_R + F' P_F. \tag{48}$$

We shall now discuss the boundary action in more detail.

C. Boundary action

The following considerations are in analogy to Sec. II.D in [15]. Because varying N_+ would lead to zero ADM mass and varying N_0 would restrict F_0 to zero, both N_+ and N_0

should be considered as prescribed functions. By the fall-off conditions, the lapse function, N^r , is required to vanish both at the center as well as at infinity. This implies that the time evolution is generated along the world lines of observers with $r = \text{constant}$. If we introduce the proper time of these observers as a new variable, we can express the lapse function in the form $N_+(t) = \dot{\bar{\tau}}_+$ and $N_0(t) = \dot{\bar{\tau}}_0$. This leads to

$$S_{\partial\Sigma} = - \int dt M_+(t) \dot{\bar{\tau}}_+ + \int dt M_0(t) \dot{\bar{\tau}}_0. \quad (49)$$

Thus we remove the need to fix the lapse function at the boundaries. Extending the treatment in [12], the aim is to cast the homogeneous part of the action into Liouville form, finding a transformation to new variables that absorb the boundary terms. This can be done by introducing the mass density $\Gamma = F'$ as a new canonical variable. Define

$$F(r) = M_0 + \int_0^r dr' \Gamma(r'), \quad \Gamma(r) = F'(r), \quad (50)$$

and reconsider the Liouville form

$$\begin{aligned} \Theta &:= \int_0^\infty dr P_F \delta F - M_+ \delta \bar{\tau}_+ + M_0 \delta \bar{\tau}_0 \\ &= \int_0^\infty P_F \delta F + \bar{\tau}_+ \delta M_+ - \bar{\tau}_0 \delta M_0, \end{aligned} \quad (51)$$

where we have dropped an exact form. But

$$\delta F = \delta M_0 + \int_0^r dr' \delta \Gamma(r') \quad (52)$$

gives

$$\Theta = \left(\int_0^\infty dr' P_F(r') - \bar{\tau}_0 \right) \delta M_0 + \int_0^\infty dr P_F(r) \int_0^r dr' \delta \Gamma(r') + \bar{\tau}_+ \delta M_+. \quad (53)$$

Noting further that⁵

$$\int_0^\infty dr P_F(r) \int_0^r dr' \delta \Gamma(r') = \int_0^\infty dr \delta \Gamma(r) \int_r^\infty dr' P_F(r'), \quad (54)$$

⁵ See Kuchař [12]. Consider

$$\begin{aligned} &\left(\int_0^r dr' \delta \Gamma(r') \times \int_r^\infty dr' P_F(r') \right)' \\ &= \delta \Gamma(r) \times \int_r^\infty dr' P_F(r') - P_F(r) \times \int_0^r dr' \delta \Gamma(r') \end{aligned}$$

Integrating the left hand side from 0 to ∞ gives zero, therefore

$$\int_0^\infty dr P_F(r) \int_0^r dr' \delta \Gamma(r') = \int_0^\infty dr \delta \Gamma(r) \int_r^\infty dr' P_F(r')$$

we can write the Liouville form as

$$\begin{aligned}
\Theta &= \left(\int_0^\infty dr' P_F(r') - \bar{\tau}_0 \right) \delta M_0 \\
&\quad + \int_0^\infty dr \delta \Gamma(r) \left(\int_0^\infty dr P_F(r) - \int_0^r dr' P_F(r') \right) + \bar{\tau}_+ \delta M_+ \\
&= \left(\int_0^\infty dr' P_F(r') - \bar{\tau}_0 \right) \delta M_0 \\
&\quad + (\delta M_+ - \delta M_0) \int_0^\infty dr P_F(r) - \int_0^\infty dr \delta \Gamma(r) \int_0^r dr' P_F(r') + \bar{\tau}_+ \delta M_+ \\
&= p_0 \delta M_0 + p_+ \delta M_+ + \int_0^\infty P_\Gamma(r) \delta \Gamma(r),
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
p_0 &= -\bar{\tau}_0, \\
p_+ &= \bar{\tau}_+ + \int_0^\infty dr P_F(r), \\
P_\Gamma(r) &= - \int_0^r dr' P_F(r').
\end{aligned} \tag{56}$$

The new form of the action is then

$$S_{\text{EH}} = \int dt \left(p_0 \dot{M}_0 + p_+ \dot{M}_+ + \int dr [P_\tau \dot{\tau} + \bar{P}_R \dot{R} + P_\Gamma \dot{\Gamma} - N \mathcal{H}^g - N^r \mathcal{H}_r] \right), \tag{57}$$

where the new constraints read

$$\begin{aligned}
\mathcal{H}^g &= -\frac{1}{L} [-\mathcal{F} P'_\Gamma \bar{P}_R + \mathcal{F}^{-1} R' \Gamma] + P_\tau \sqrt{1 + \frac{\tau'^2}{L^2}} \approx 0, \\
\mathcal{H}_r &= R' \bar{P}_R - \Gamma P'_\Gamma + \tau' P_\tau \approx 0.
\end{aligned} \tag{58}$$

The Hamiltonian constraint can be simplified if the momentum constraint is used to eliminate P_Γ , and the constraints in (58) can be replaced by the following equivalent set

$$\begin{aligned}
P_\tau^2 + \mathcal{F} \bar{P}_R^2 - \frac{\Gamma^2}{\mathcal{F}} &\approx 0 \\
\mathcal{H}_r &= \tau' P_\tau + R' \bar{P}_R - \Gamma P'_\Gamma \approx 0.
\end{aligned} \tag{59}$$

(Note that the Hamiltonian constraint has been ‘squared’ in order to arrive at the new form; it is therefore of dimension mass over length squared. Re-inserting G would correspond to the substitution $\Gamma \rightarrow \Gamma/G$.) This involves a little algebra which has been described in the appendix of [15]. These equations will be used for quantization in the next section.

Here we emphasize that the relative sign between the dust and gravitational kinetic terms can change because \mathcal{F} is greater than zero outside the horizon and less than zero inside. This

change of sign is already present in (58) and has not been introduced by using the momentum constraint to eliminate P_F . It is of fundamental interest in the quantum theory because the Wheeler–DeWitt equation becomes locally elliptic outside and hyperbolic inside the horizon, which is of importance for the formulation of the proper boundary value problem. A change of sign was also noted for the case of a non-minimally coupled scalar field [26], see also [27] for an earlier discussion in the black-hole case.

D. Relation between dust proper time and Killing time

In what follows, we will have need of the relationship between the dust proper time and Killing time; therefore we now address this issue. This subsection corresponds to subsection II.E in the LTB case [15]. We had obtained the expression for τ' in terms of the canonical variables in Sec. IIA as

$$\tau' = P'_R \sqrt{2(E - F)} \mp \frac{R'}{\mathcal{F}} \sqrt{2(E - F) - \mathcal{F}} \quad (60)$$

Defining $a = 1/\sqrt{2(E - F)}$ gives

$$\tau' = \frac{P'_R}{a} \mp \frac{R'}{a\mathcal{F}} \sqrt{1 - a^2\mathcal{F}}. \quad (61)$$

(The equation of motion guarantees that the quantity $\sqrt{1 - a^2\mathcal{F}}$ is real.) For a constant value of a , this equation can be integrated. When $R > \sqrt{2F/\Lambda}$,

$$\begin{aligned} a\tau &= P_R \mp \int \frac{dR}{\mathcal{F}} \sqrt{1 - a^2\mathcal{F}} \\ &= P_R \pm \left[\frac{a}{\sqrt{\Lambda}} \tan^{-1} \frac{aR\sqrt{\Lambda}}{\sqrt{1 - a^2\mathcal{F}}} + \frac{1}{\sqrt{2\Lambda F}} \tanh^{-1} \frac{\sqrt{1 - a^2\mathcal{F}}}{R\sqrt{\Lambda/2F}} \right] \end{aligned} \quad (62)$$

and

$$a\tau = P_R \pm \left[\frac{a}{\sqrt{\Lambda}} \tan^{-1} \frac{aR\sqrt{\Lambda}}{\sqrt{1 - a^2\mathcal{F}}} + \frac{1}{\sqrt{2\Lambda F}} \tanh^{-1} \frac{R\sqrt{\Lambda/2F}}{\sqrt{1 - a^2\mathcal{F}}} \right] \quad (63)$$

when $R < \sqrt{2F/\Lambda}$. We know that the spacetime surrounding a collapsing cloud in 2+1 dimensions with negative cosmological constant is the BTZ black hole. The choice of sign depends on whether one is interested in an expanding (positive sign) or collapsing (negative sign) cloud. Matching the collapse and BTZ metrics also shows that $T = P_R$ at the boundary.

Equations (62) and (63) may still be used so long as E' and F' are sufficiently small, since then we have a small amount of dust propagating in the BTZ background. In that case, it would give the relationship between the time used by families of freely falling observers and the Killing time, each family being characterized by a fixed value of E .

E. Interpretation of the canonical data

We have already reconstructed the mass function from the canonical data. In this section we want to reconstruct the energy function E and singularity curve τ_0 from the same. The

three functions E , F , and τ_0 determine the collapse model completely, so reconstructing these quantities gives physical meaning to the canonical variables.

If we begin with the momentum constraint in the form

$$\tau' = -\frac{R'\bar{P}_R}{P_\tau} + \frac{\Gamma P'_\Gamma}{P_\tau} \quad (64)$$

and use the Hamiltonian constraint to eliminate \bar{P}_R , we get

$$\tau' = \pm \frac{R'}{\mathcal{F}} \sqrt{\frac{\Gamma^2}{P_\tau^2} + \mathcal{F}} + \frac{\Gamma P'_\Gamma}{P_\tau}. \quad (65)$$

Substituting $P'_\Gamma = -P_F = LP_L/\mathcal{F}$, we find an expression,

$$\tau' = \pm \frac{R'}{\mathcal{F}} \sqrt{\frac{\Gamma^2}{P_\tau^2} + \mathcal{F}} + \frac{\Gamma LP_L}{\mathcal{F} P_\tau} \quad (66)$$

that may be directly compared with the expression we had in (28) for τ' . We see that

$$P_\tau = \frac{\Gamma}{\sqrt{2(E-F)}}, \quad (67)$$

and therefore the energy function is related to the canonical variables by

$$E = \frac{\Gamma^2}{2(\Gamma^2/\mathcal{F} - \mathcal{F}\bar{P}_R^2)} + F \quad (68)$$

where we have used the Hamiltonian constraint to give the result in terms of gravitational phase space variables only. Finally, knowing E and F in terms of the canonical variables, it is a simple matter to do the same for the singularity curve using the solution (7), cf. [15].

F. Hamilton equations of motion

We now give the Hamilton equations of motion for the new system and derive Einstein's equations (3) from them. Introducing the smeared constraints

$$\begin{aligned} H[N] &= \int_0^\infty dr N(r) \mathcal{H}(r), \\ H_r[N^r] &= \int_0^\infty dr N^r(r) \mathcal{H}_r(r) \end{aligned} \quad (69)$$

and the Hamiltonian $\mathbb{H} = \mathcal{H}[N] + \mathcal{H}_r[N^r]$, the canonical equations of the evolution are⁶

$$\dot{M}_0 = \frac{\delta \mathbb{H}}{\delta p_0} = 0,$$

⁶ Note that

$$\frac{\delta F(r')}{\delta \Gamma(r)} = \Theta(r' - r),$$

where Θ is the step function.

$$\begin{aligned}
\dot{p}_0 &= -\frac{\delta \mathbb{H}}{\delta M_0} = 2 \int_0^\infty dr \, N \left(\overline{P}_R^2 + \frac{\Gamma^2}{\mathcal{F}^2} \right), \\
\dot{M}_+ &= \frac{\delta \mathbb{H}}{\delta p_+} = 0, \\
\dot{p}_+ &= -\frac{\delta \mathbb{H}}{\delta M_+} = 0, \\
\dot{\tau} &= \frac{\delta \mathbb{H}}{\delta P_\tau} = 2NP_\tau + N^r \tau', \\
\dot{P}_\tau &= -\frac{\delta \mathbb{H}}{\delta \tau} = (N^r P_\tau)', \\
\dot{R} &= \frac{\delta \mathbb{H}}{\delta \overline{P}_R} = -2N\mathcal{F}\overline{P}_R + N^r R', \\
\dot{\overline{P}}_R &= -\frac{\delta \mathbb{H}}{\delta R} = 2N \left(\Lambda R \overline{P}_R^2 + \frac{\Lambda R \Gamma^2}{\mathcal{F}^2} \right) + (N^r \overline{P}_R)', \\
\dot{\Gamma} &= \frac{\delta \mathbb{H}}{\delta P_\Gamma} = (N^r \Gamma)', \\
\dot{P}_\Gamma &= -\frac{\delta \mathbb{H}}{\delta \Gamma} = \frac{2N\Gamma}{\mathcal{F}}, \\
&\quad + 2 \int_0^r dr' N(r') \left[\overline{P}_R(r')^2 + \frac{\Gamma(r')^2}{\mathcal{F}(r')^2} \right] + N^r P'_\Gamma.
\end{aligned} \tag{70}$$

To obtain Einstein's equations, we notice that because

$$R^* = \frac{\dot{R}}{\dot{\tau}} \stackrel{N^r=0}{=} -\frac{\mathcal{F}\overline{P}_R\sqrt{2(E-F)}}{\Gamma}, \tag{71}$$

the momentum \overline{P}_R conjugate to R may be expressed as

$$\overline{P}_R = -\frac{\Gamma R^*}{\mathcal{F}\sqrt{2(E-F)}}. \tag{72}$$

Substituting this result in the Hamiltonian constraint leads directly to (3). Note that we did not have to specify the lapse function.

The lapse no longer has the interpretation of being the ratio between the proper time to BTZ time. The reason is that we have squared the original version of the Hamiltonian constraint. If we define a new version of the constraint by taking the square root,

$$\mathcal{H}_\uparrow = P_\tau - \sqrt{-\mathcal{F}\overline{P}_R^2 + \frac{\Gamma^2}{\mathcal{F}}}, \tag{73}$$

then

$$\{\tau, \mathcal{H}_\uparrow[N^\uparrow]\} = N^\uparrow \tag{74}$$

shows that it is N^\uparrow that carries that interpretation.

Finally, we remark that the algebra of the constraints cannot be of the general form (given for example in [1]), again because we have used the momentum constraint to eliminate P_F in the Hamiltonian constraint. In fact, a short calculation gives

$$\{\mathcal{H}[N], \mathcal{H}[M]\} = 0, \tag{75}$$

$$\{\mathcal{H}_r[N^r], \mathcal{H}[N]\} = \mathcal{H}[N_{,r}N^r - NN^r_{,r}], \tag{76}$$

$$\{\mathcal{H}_r[N^r], \mathcal{H}_r[M^r]\} = \mathcal{H}_r[[N^r, M^r]]. \tag{77}$$

The Poisson bracket of the Hamiltonian with itself vanishes, in contrast to the general case where it closes on the momentum constraint. The other brackets coincide with the general case. The transformations generated by the Hamiltonian constraint can thus no longer be interpreted as hypersurface deformations. They are in general not orthogonal to the hypersurfaces, but act along the dust flow lines.

IV. EXACT QUANTUM STATES

A. Quantization and ansatz

We now apply Dirac's quantization procedure to turn the classical constraints into quantum operators

$$P_X = -i \frac{\delta}{\delta X(r)}, \quad (78)$$

and acting the operators on wave-functionals. The Hamiltonian constraint then reads

$$\left[\frac{\delta^2}{\delta \tau^2} + \mathcal{F} \frac{\delta^2}{\delta R^2} + A \delta(0) \frac{\delta}{\delta R} + B \delta(0)^2 - \frac{\Gamma^2}{\mathcal{F}} \right] \Psi[\tau, R, \Gamma] = 0. \quad (79)$$

where A and B are smooth functions of R and F that encapsulate the factor ordering ambiguities. The factor ordering problem is unsolved and can be dealt with only after a suitable regularization procedure is implemented. We have included the formal expression $\delta(0)$ to indicate the need for this regularization. Quantizing the momentum constraint using (78) gives

$$\left[\tau' \frac{\delta}{\delta \tau} + R' \frac{\delta}{\delta R} - \Gamma \left(\frac{\delta}{\delta \Gamma} \right)' \right] \Psi[\tau, R, \Gamma] = 0, \quad (80)$$

but, as noted, the quantum constraints in (79) and (80) are only formal until a regularization procedure has been selected.

To regularize, we follow the construction [15] and consider a one dimensional lattice of discrete points r_i separated by a distance σ which must be later taken to zero to achieve the continuum limit. We will require the wave-functional to (i) automatically satisfy the momentum constraint in the continuum limit and (ii) be factorizable into different functions for each lattice point. Such a wave functional is of the form

$$\begin{aligned} \Psi[\tau, R, \Gamma] &= \exp \left[i \int dr \Gamma(r) \mathcal{W}(\tau(r), R(r), F(r)) \right] \\ &= \lim_{\sigma \rightarrow 0} \prod_i \exp [i \sigma \Gamma_i \mathcal{W}_i(\tau_i, R_i, F_i)], \end{aligned} \quad (81)$$

where $F_i = \sum_{j=0}^i \sigma \Gamma_j$. Details of the construction are given in [15]. One finds that the Hamiltonian constraint is satisfied independently of the choice of σ only if the following three equations are simultaneously obeyed:

$$\left[\left(\frac{\partial \mathcal{W}_j}{\partial \tau_j} \right)^2 + \mathcal{F}_j \left(\frac{\partial \mathcal{W}_j}{\partial R_j} \right)^2 - \frac{1}{\mathcal{F}_j} \right] = 0,$$

$$\left[\frac{\partial^2 \mathcal{W}_j}{\partial \tau_j^2} + \mathcal{F}_j \frac{\partial^2 \mathcal{W}_j}{\partial R_j^2} + A_j \frac{\partial \mathcal{W}_j}{\partial R_j} \right] = 0, \quad (82)$$

$$B_j = 0.$$

The first equation is the Hamilton–Jacobi equation and was used in [13]. The second equation presents an additional restriction on solutions and the last equation tells us that working on the lattice is only possible if the factor ordering does not contribute to the potential term. If we find solutions to all three equations, we can do all other calculations on the lattice, since these solutions have a well defined continuum limit and satisfy the momentum constraint. We will now find exact solutions to all three equations.

B. The measure

The function A is closely tied to the inner product on the Hilbert space via the hermiticity of the operator

$$\widehat{\mathcal{F}_j P_{j,R}^2} = \mathcal{F}_j \frac{\partial^2}{\partial R_j^2} + A_j \frac{\partial}{\partial R_j}. \quad (83)$$

It is therefore intimately connected to the choice of measure, μ_j . Below we consider the relationship between μ and A , assuming that μ , is independent of τ . For $\widehat{O} = \widehat{\mathcal{F}P_R^2}$ to be hermitian, we require

$$\int_0^\infty dR \mu(R) \phi^*(R) \{ \widehat{O} \psi(R) \} = \int_0^\infty dR \mu(R) \{ \widehat{O} \phi(R) \}^* \psi(R), \quad (84)$$

which leads, provided the boundary conditions are trivial, to the (coupled) system

$$A_j^* = -A_j + \frac{2}{\mu_j} \frac{\partial}{\partial R_j} (\mu_j \mathcal{F}_j),$$

$$\frac{\partial^2}{\partial R_j^2} (\mu_j \mathcal{F}_j) - \frac{\partial}{\partial R_j} (\mu_j A_j) = 0. \quad (85)$$

The second equation can be integrated to give

$$\frac{1}{\mu_j} \frac{\partial}{\partial R_j} (\mu_j \mathcal{F}_j) - \frac{\mathcal{Q}_j(F_j)}{\mu_j} = A_j, \quad (86)$$

and the first requires that $\Re(\mathcal{Q}) = 0$. We can view (86) as a (Bernoulli) equation for μ_j :

$$\frac{\partial \mu_j}{\partial R_j} + \frac{1}{\mathcal{F}_j} \left(\frac{\partial \mathcal{F}_j}{\partial R_j} - A_j \right) \mu_j = \frac{\mathcal{Q}_j(F_j)}{\mathcal{F}_j}, \quad (87)$$

whose general solution is

$$\mu_j = e^{-\int \mathcal{P}_j(R_j) dR_j} \left[\pm \mathcal{Q}_j(F_j) \int dR_j \frac{e^{\int \mathcal{P}_j(R_j) dR_j}}{|\mathcal{F}_j|} + \alpha_j \right], \quad (88)$$

where the upper sign is for the exterior ($\mathcal{F} > 0$), the lower sign is for the interior and

$$\mathcal{P}_j = \frac{1}{|\mathcal{F}_j|} \left(\frac{\partial |\mathcal{F}_j|}{\partial R_j} \mp A_j \right), \quad (89)$$

again with the same sign conventions.

For solutions of (82), discussed in the following subsection, the relation between A and μ ,

$$A_j = |\mathcal{F}_j| \partial_{R_j} \ln(\mu_j |\mathcal{F}_j|), \quad (90)$$

which re-expresses (86) with $\mathcal{Q}_j(F_j) = 0$ (because $\mathcal{Q}_j(F_j)$ can only be imaginary), will be of greater interest than (88).

C. Exact solutions

We have already emphasized that the signature in the kinetic part of the Hamiltonian constraint can change from elliptic (outside the horizon) to hyperbolic (inside the horizon). This carries over to the kinetic term of the Wheeler-DeWitt equation. As discussed in [27], we can say that the part inside the horizon is always classically allowed, whereas this is not necessarily the case for the outside part. The usual initial value problem appropriate for hyperbolic equations can thus only be applied for the region corresponding to the black hole interior.

We determine the unique, exact solutions to (82) in Appendix A and here we summarize the results. Both in the exterior, $\mathcal{F} > 0$, and interior, $\mathcal{F} < 0$, we obtain a two parameter family of solutions: for the measure

$$\mu_j = \frac{\beta_j}{\sqrt{1 - a_j^2 \mathcal{F}_j}}, \quad (91)$$

and for \mathcal{W}_j

$$\mathcal{W}_j = \text{const.} \pm a_j \tau_j \pm \int dR_j \frac{\sqrt{1 - a_j^2 \mathcal{F}_j}}{\mathcal{F}_j}, \quad (92)$$

where a_j and β_j are constants of integration. The physical significance of a_j can be demonstrated by acting on our wave functional with the dust energy operator \hat{P}_τ :

$$\hat{P}_\tau \Psi_a = a \Gamma \Psi_a = \frac{\Gamma}{\sqrt{2(E - F)}} \Psi_a, \quad (93)$$

showing that $a = 1/\sqrt{2(E - F)}$. The integral appearing in (92) will be recognized then as the integral we evaluated earlier in (63) in order to obtain the relationship between the dust proper time and the BTZ time. The wave functionals are therefore oscillatory in the classically allowed regions.

Given the requirement of factorizability on the lattice, there are no other solutions (either separating or non-separating) that solve the constraints, that is, we have obtained the complete class of factorizable solutions to all the constraints. Other solutions would necessarily

couple the shells composing the collapsing cloud. The solutions we have obtained, which are strictly valid on the lattice, contain two free parameters, a_j and b_j , and we write them as

$$\Psi_j = e^{-\sigma b_j \Gamma_j} e^{\pm i \sigma \Gamma_j (a_j \tau_j \pm \int dR_j \frac{\sqrt{1-a_j^2 \mathcal{F}_j}}{\mathcal{F}_j})}. \quad (94)$$

We know that a_j is connected with the energy and mass functions and that in general both E and F are functions of the radial label coordinate. It would therefore be natural to demand that a is also an arbitrary function of r , but this explicit dependence on r would violate the momentum constraint. The way out is to require the r dependence of a to appear via the mass function, $F(r)$, *i.e.*, we must require that $a = a(F(r))$, and likewise $b = b(F(r))$. Any dependence on r via the mass function is allowed by the momentum constraint and since the Hamiltonian constraint does not contain a derivative with respect to Γ or F it continues to be obeyed. Hence, in the continuum limit we arrive at

$$\Psi[\tau, R, \Gamma] = e^{-\int_0^\infty dr \Gamma(r) b(F(r))} \exp \left\{ \pm i \int_0^\infty dr \Gamma(r) \left[a(F(r)) \tau \pm \int dR \frac{\sqrt{1-a^2(F(r)) \mathcal{F}}}{\mathcal{F}} \right] \right\}. \quad (95)$$

The fact that $a(r)$ is constrained by diffeomorphism invariance to depend on r only via $F(r)$ means that the energy function is not arbitrary in the quantum theory but only those energy functions expressible in terms of F , *i.e.*, $E(r) = F(r) + 1/2a^2(F(r))$ are allowed.

V. HAWKING RADIATION

A. Introduction

The exact quantum states in (95) describe the generic situation. In order to describe Hawking radiation, we need to introduce into the formalism the black hole and the analogue of the quantum fields which are used in the standard treatment.

Following [13] and [16], we consider a BTZ black hole surrounded by tenuous dust, by taking the mass function to be of the form $2F(r) = GM\Theta(r) + 2f(r)$, where $\Theta(r)$ is the Heaviside function and $2f(r)$ represents the dust perturbation. Formally, the variable $\Gamma = GM\delta(r) + f'(r)$ describes this perturbation and $f'(r)$ describes the mass density outside $r = 0$. As described in [13], the black hole state factors into a product state, one member of the which represents the black hole, while the other represents the dust and describes the Hawking radiation.

The central idea of [13] was to use the Bogoliubov transformation of the field operators in the BTZ spacetime. Here, since we have the exact quantum states at our disposal, we first identify the states that correspond to the ingoing and outgoing modes, respectively, of the standard approach and then calculate their inner product. The inner product is evaluated on hypersurfaces of constant Killing time, which correspond to observers who are static with respect to the black hole, not constant dust time, which would correspond to observers who are freely falling. We will therefore express our solutions in terms of the BTZ time, T , using (63).

The relevant exact solutions of the Wheeler–DeWitt equation read (in this section we re-introduce \hbar and G)

$$\Psi^\pm = \exp \left(-\frac{1}{G\hbar} \int dr b \Gamma \mp \frac{i}{2G\hbar} \int dr \Gamma \left[a\tau \pm \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} \right] \right) . \quad (96)$$

We will not concern ourselves with the normalizations at this point because they make no difference to our final result.

For the positive frequency, incoming wave functional we choose Ψ^+ from (96) and replace the dust proper time with the Killing time using (63). We find

$$\Psi^+ = \exp \left(-\frac{1}{G\hbar} \int dr b \Gamma - \frac{i}{G\hbar} \int dr \Gamma T \right) , \quad (97)$$

which has the standard form of a positive-frequency wave function. This wave-functional is independent of R . The lattice version of this state is obtained through the replacement of $\Gamma = F'$ by a frequency ω , and is given by

$$\Psi_{\sigma\omega}^+ = \lim_{\sigma \rightarrow 0} \prod_j e^{(-\sigma b\omega_j - i\sigma\omega_j T_j)/G\hbar} . \quad (98)$$

For the out-going modes of negative frequency we have to take the state Ψ^- . Inserting (63) into the corresponding state of (96) then gives

$$\Psi^- = \exp \left(-\frac{1}{G\hbar} \int dr b \Gamma + \frac{i}{G\hbar} \int dr \Gamma \left[T - 2 \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} \right] \right) , \quad (99)$$

the corresponding lattice version being

$$\Psi_{\sigma\omega}^- = \lim_{\sigma \rightarrow 0} \prod_j \exp \left(-\frac{\sigma b_j \omega_j}{G\hbar} + \frac{i\sigma\omega_j}{G\hbar} \left[T_j - 2 \int dR_j \frac{\sqrt{1-a_j^2\mathcal{F}_j}}{\mathcal{F}_j} \right] \right) . \quad (100)$$

B. The Bogoliubov coefficient in the near horizon limit

The Bogoliubov coefficients [16, 28] are given by

$$\beta_{\omega\omega'} = \frac{2\sigma\omega}{G\hbar} \int_{R_h}^{\infty} dR \sqrt{g_{RR}} \Psi_{\sigma\omega}^{-*} \Psi_{\sigma\omega'}^+ . \quad (101)$$

Using the coordinate transformation from (R, τ) to (R, T) , we have in (R, T) coordinates,

$$\sqrt{g_{RR}} = \frac{1}{a(\Lambda R^2 - 2F)} , \quad (102)$$

where $a = \frac{1}{\sqrt{2(E-F)}}$. Inserting the wave functionals (98) and (100) into (101) then gives

$$\begin{aligned} \beta_{\omega\omega'} &= \frac{2\omega\sigma}{G\hbar} \sqrt{2(E-F)} \exp \left(-\frac{\sigma b(\omega + \omega')}{G\hbar} - i\frac{\sigma T(\omega + \omega')}{G\hbar} \right) \times \\ &\int_F^{\infty} dR \frac{1}{\Lambda R^2 - 2F} \exp \left[\frac{2i\sigma\omega}{G\hbar} \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} \right] . \end{aligned} \quad (103)$$

We now change the variable of integration to \mathcal{F} , keeping in mind the near horizon approximation. \mathcal{F} goes to zero on the horizon and diverges as R goes to infinity. We have $d\mathcal{F}/2\Lambda R = dR$. Rewriting the integral in the new variable yields

$$\beta_{\omega\omega'} = \frac{2\omega\sigma}{G\hbar} \sqrt{2(E-F)} \exp\left(-\frac{\sigma b(\omega+\omega')}{G\hbar} - i\frac{\sigma T(\omega+\omega')}{G\hbar}\right) \times \int_0^\infty d\mathcal{F} \frac{1}{2\mathcal{F}\sqrt{\Lambda(\mathcal{F}+2F)}} \exp\left[\frac{2i\sigma\omega}{G\hbar} \left(\int^\mathcal{F} d\tilde{\mathcal{F}} \frac{\sqrt{(1-a^2\tilde{\mathcal{F}})}}{2\tilde{\mathcal{F}}\sqrt{\Lambda(\tilde{\mathcal{F}}+2F)}}\right)\right]. \quad (104)$$

In the near horizon limit, we expand about $\mathcal{F} = 0$. The integral becomes

$$\beta_{\omega\omega'} = \frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \sqrt{2(E-F)} \exp\left(-\frac{\sigma b(\omega+\omega')}{G\hbar} - i\frac{\sigma T(\omega+\omega')}{G\hbar}\right) \times \int_0^\infty d\mathcal{F} \mathcal{F}^{-1+\frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}} \exp\left[-i\frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \left(\frac{a^2}{2} + \frac{1}{4F}\right) \mathcal{F}\right]. \quad (105)$$

Using the following formula after inserting the regularization factor $e^{-p\mathcal{F}}$ for convergence,

$$\int_0^\infty dx x^{\nu-1} e^{-(p+iq)x} = \Gamma(\nu)(p^2+q^2)^{-\nu/2} e^{-i\nu\arctan(q/p)},$$

where $\nu = \frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}$ and $q = \frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \left(\frac{a^2}{2} + \frac{1}{4F}\right) \mathcal{F}$, we find

$$\beta_{\omega\omega'} = \frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \sqrt{2(E-F)} \exp\left(-\frac{\sigma b(\omega+\omega')}{G\hbar} - i\frac{\sigma T(\omega+\omega')}{G\hbar}\right) \times \Gamma\left(\frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\right) \left(\frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \left(\frac{a^2}{2} + \frac{1}{4F}\right)\right)^{-i\sigma\omega/(G\hbar\sqrt{2\Lambda F})} e^{-\pi\sigma\omega/(2G\hbar\sqrt{2\Lambda F})}. \quad (106)$$

The absolute square of the above expression is given by

$$|\beta_{\omega\omega'}|^2 = \frac{2\pi\sigma\omega(2(E-F))}{G\hbar\sqrt{2\Lambda F}} \frac{e^{-2\sigma b(\omega+\omega')/G\hbar}}{e^{2\pi\sigma\omega/G\hbar\sqrt{2\Lambda F}} - 1} \quad (107)$$

and determines the particle creation via

$$\langle \text{in} | \hat{N}_{\text{out}} | \text{in} \rangle = \int_0^\infty d(\sigma\omega') |\beta_{\omega\omega'}|^2. \quad (108)$$

(We integrate here over $\sigma\omega'$ in order to obtain a dimensionless expression.) Performing the integration and replacing $\sigma\omega$ by $G\Delta\epsilon$, where $\Delta\epsilon$ denotes the energy of a shell, we get (setting also $2F = GM$)

$$\langle \text{in} | \hat{N}_{\text{out}} | \text{in} \rangle = \frac{\pi\Delta\epsilon(2E-GM)}{b\sqrt{\Lambda GM}} \frac{e^{-2b\Delta\epsilon/\hbar}}{e^{2\pi\Delta\epsilon/\hbar\sqrt{\Lambda GM}} - 1}, \quad (109)$$

where \hat{N}_{out} represents the “out” particle number.

We recognize that the Planck spectrum is modified by greybody factors which explicitly depend on the energy. These greybody factors are different from those obtained by taking

into account back-scattering (for example, see [29]) because we are using exact solutions of the quantum constraints. Both the gravitational field and the dust are quantized in our approach.

The constant b that appears in the final expression for $\langle \text{in} | N_{\text{out}} | \text{in} \rangle$ also has its origin in the full quantum gravitational state. The simplest choice for b is $b = 0$, in which case the integral over ω' diverges. This divergence, however, is well-known and is connected with the normalization of the continuous modes. Dividing out the infinite constant from this integration the result, for $b = 0$, is⁷

$$\langle \text{in} | \hat{N}_{\text{out}} | \text{in} \rangle = \frac{\pi \Delta \epsilon (2E - GM)}{\hbar \sqrt{\Lambda GM}} \frac{1}{e^{2\pi \Delta \epsilon / \hbar \sqrt{\Lambda GM}} - 1} \quad (110)$$

This is a thermal spectrum with temperature given by

$$k_B T_H = \hbar \frac{\sqrt{\Lambda GM}}{2\pi}, \quad (111)$$

which holds for each shell separately.

C. Exact Bogoliubov coefficient

It is possible to obtain an exact expression for the Bogoliubov coefficient as well by considering a series expansion about the horizon. We expand equation (104) around $\mathcal{F} = 0$ to all orders. Expanding the measure of equation (104), we get,

$$\frac{\sqrt{1 + \frac{\mathcal{F}}{2F}}}{2\sqrt{\Lambda 2F}\mathcal{F}} = \frac{1}{2\sqrt{\Lambda 2F}} \frac{1}{\mathcal{F}} \left(1 - \frac{\mathcal{F}}{4F} + \frac{3}{8} \left(\frac{\mathcal{F}}{2F} \right)^2 \dots \right) \stackrel{\text{def}}{=} \sum_{n=-1}^{\infty} \alpha_n s^n, \quad (112)$$

Similarly the exponent can be rewritten as,

$$\begin{aligned} & \frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \left(\ln \mathcal{F} - \left(\frac{a^2}{2} + \frac{1}{4F} \right) \mathcal{F} + \dots \right) \\ & \stackrel{\text{def}}{=} \frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \left(\ln \mathcal{F} + \sum_{m=1}^{\infty} \beta_m \mathcal{F}^m \right), \end{aligned} \quad (113)$$

The integral can be performed before summing over the series of (112).

$$\begin{aligned} & \frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}} \sqrt{2(E - F)} \times \\ & \sum_{n=-1}^{\infty} \alpha_n \int_0^{\infty} d\mathcal{F} \mathcal{F}^{n+i\sigma\omega/(G\hbar\sqrt{2\Lambda F})} e^{i\sigma\omega\beta_1\mathcal{F}/(G\hbar\sqrt{2\Lambda F})} e^{i\sigma\omega \sum_{m=2}^{\infty} \beta_m \mathcal{F}^m / (G\hbar\sqrt{2\Lambda F})} \end{aligned} \quad (114)$$

⁷ We divide by $G\hbar$ in order to keep the result dimensionless.

where $\beta_1 = -(1/a^2 + 1/4F)$.

$$e^{i\frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\sum_{m=2}^{\infty}\beta_m\mathcal{F}^m} = 1 + \frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\sum_{m=2}^{\infty}\beta_m\mathcal{F}^m + \frac{1}{2}\left(\frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\right)^2\sum_{m,n=2}^{\infty}\beta_m\beta_n\mathcal{F}^m\mathcal{F}^n \equiv \sum_{m=0}^{\infty}\gamma_m\mathcal{F}^m, \quad (115)$$

where $\gamma_0 = 1$ and $\gamma_1 = 0$. The integral is

$$\begin{aligned} &= \frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\sqrt{2(E-F)}\sum_{n,m=0}^{\infty}\alpha_{n-1}\gamma_m\int_0^{\infty}d\mathcal{F}\mathcal{F}^{-1+n+m+i\sigma\omega/(G\hbar\sqrt{2\Lambda F})}e^{i\sigma\omega\beta_1\mathcal{F}/(G\hbar\sqrt{2\Lambda F})} \\ &= \frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\sqrt{2(E-F)}\times \\ &\quad \sum_{n,m=0}^{\infty}\alpha_{n-1}\gamma_m\left(-i\frac{\partial}{\partial\chi}\right)^{n+m}\int_0^{\infty}d\mathcal{F}\mathcal{F}^{-1+i\sigma\omega F/(G\hbar\sqrt{2\Lambda F})}e^{i\chi\mathcal{F}}\Bigg|_{\chi=\sigma\omega\beta_1/(G\hbar\sqrt{2\Lambda F})} \end{aligned} \quad (116)$$

and the exact Bogoliubov coefficient can be written as

$$\begin{aligned} \beta_{\omega\omega'} &= \frac{\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\sqrt{2(E-F)}\exp\left(-i\frac{\sigma T(\omega+\omega')}{G\hbar}\right)\times \\ &\quad \left(\frac{\sigma\omega\beta_1}{G\hbar\sqrt{2\Lambda F}}\right)^{-i\sigma\omega/(G\hbar\sqrt{2\Lambda F})}e^{-\pi\sigma\omega/(2G\hbar\sqrt{2\Lambda F})}\Gamma\left(\frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\right)\times [1+ \\ &\quad \sum_{n+m=1}^{\infty}(-i)^{n+m}\alpha_{n-1}\gamma_m\left(-\frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\right)\times\cdots\times\left(-\frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}-n-m+1\right)\times \\ &\quad \left(\frac{\sigma\omega\beta_1}{G\hbar\sqrt{2\Lambda F}}\right)^{-n-m}]. \end{aligned} \quad (117)$$

The simplest correction term to the near-horizon approximation is ω -independent and obtained for $n = 1, m = 0$:

$$(-i)\alpha_0\gamma_0\left(-\frac{i\sigma\omega}{G\hbar\sqrt{2\Lambda F}}\right)\left(\frac{\sigma\omega\beta_1}{G\hbar\sqrt{2\Lambda F}}\right)^{-1} = -\frac{\alpha_0}{\beta_1}.$$

$\alpha_0 = -1/4F$ and $\beta_1 = -(a^2/2 + 1/4F)$, which can be simplified to be $-\frac{1}{2a^2E}$.

In general the higher order corrections are ω dependent and therefore lead to non-trivial modifications of the greybody factors. However, in the limit as $\omega \rightarrow \infty$, the corrections once again become independent of ω . These correction terms cannot be obtained in the standard derivation of the Hawking radiation because the geometric optics approximation is strictly assumed. We can obtain them because we have the exact quantum state at our disposal.

VI. DISCUSSION

In this paper we have extended the canonical formalism originally developed for the Schwarzschild black hole and later for the LTB collapse of inhomogeneous dust in 3+1

dimensions to 2+1 dimensional collapse with a negative cosmological constant. In this work, our emphasis has been on the physical parameters E , F , and τ_0 , and we have succeeded in expressing them in terms of the canonical variables. This renders the formalism more transparent.

Using a lattice regularization scheme that correctly implements the diffeomorphism constraint in the continuum limit and assuming factorizability on the lattice, we presented exact and unique solutions to the quantum constraints and extended them to the continuum. They coincide with the WKB approximation, which therefore become exact.

Using the canonical formulation we produced a general relationship between the dust proper time and the Killing time and then used our solutions to derive the Hawking radiation from a BTZ black hole. It was possible to reliably calculate the greybody factors because our solutions are exact. No further corrections to the Hawking radiation are obtained because the WKB solution is exact within the framework of our regularization.

It is worthwhile comparing our derivation of Hawking radiation with the corresponding calculation in the four-dimensional LTB case [16], and also with the semiclassical 2+1-d derivation of Hawking radiation [19]. There are significant differences in the gravitational collapse between the 2+1-dimensional and the 3+1-dimensional case on the classical level. The effects are indeed captured in the quantum formalism developed in the paper. Through canonical transformations both Wheeler–DeWitt equations can be transformed to the same form. The qualitative difference between the two cases is brought out by the difference in the form of \mathcal{F} which captures the information about the details of trapped surface formation. In the calculation of Hawking radiation, the framework is the same as in the four-dimensional case. In both cases it yields the correct Hawking temperature, though the correction terms for the Bogoliubov coefficients turn out to be different.

In [19], Hawking radiation is obtained semiclassically by considering the quantization of scalar field modes in the background of a BTZ spacetime. The modes are shown to have the asymptotic form which resembles the standard plane wave form with a decay factor. The behavior at spatial infinity had to be taken care of by using reflecting boundary conditions since the AdS spatial infinity is timelike. The scalar field is equated to zero at the spatial infinity. In our case, we have the wave functionals for dust. In order to evaluate the Bogoliubov coefficients, we assumed a scalar product of the form given in (101). The integral is evaluated over the spatial slice of constant Killing time. The value of the wave functional for each shell is asymptotically decaying for large R . This makes the wave functional zero at spatial boundary of the AdS. Thus our solution automatically addresses the problem of the AdS spatial infinity being timelike.

There are several outstanding issues that must remain for future work. One of particular interest is to obtain a microscopic description of the entropy of the BTZ black hole. The BTZ black hole is a special case of our solutions, corresponding to a constant mass function. Its entropy was computed using the AdS/CFT correspondence, although the physical significance of the degrees of freedom being counted in that approach remains ambiguous. An advantage of the canonical approach is that the degrees of freedom have a transparent physical meaning. A comparison between the microscopic degrees of freedom from the canonical theory and those counted using the AdS/CFT would be illuminating.

Another issue concerns the study of a possible singularity avoidance in the simplified realm of 2+1 dimensions. Singularity avoidance has been claimed from loop quantum gravity [5] and has been shown to hold for collapsing dust shells in 3+1 dimensions [30]. The quantum collapse of shells in 2+1 dimensions was recently examined in [31] and the results suggest that at least a portion of the collapsing shell rebounds, reemerging from the horizon in a finite proper time, while the remaining portion of the shell is lost to Hawking radiation.

A third issue concerns the role of naked singularities in the quantum theory. This problem is of particular experimental interest because semi-classical arguments suggest that naked singularities are quantum mechanically unstable. If those arguments hold, naked singularities could be experimental windows into the world of quantum gravity.

Finally, the fact that the WKB approximation becomes exact in the lattice regularization means that it is possibly useful to look for alternative regularization schemes. We plan to address some of these issues in future publications.

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APPENDIX : Uniqueness of solutions

In this appendix we discuss the uniqueness of the solutions in (95). We will consider the solutions in the exterior ($\mathcal{F} > 0$). The solutions in the interior are obtained in an analogous manner by replacing the trigonometric substitution below by the analogous hyperbolic one.

For $\mathcal{F} > 0$, the Hamilton-Jacobi equation [the first of (82)] is solved by taking

$$\begin{aligned}\frac{\partial \mathcal{W}_j}{\partial \tau_j} &= \frac{\cos \eta_j}{\sqrt{\mathcal{F}_j}}, \\ \frac{\partial \mathcal{W}_j}{\partial R_j} &= \frac{\sin \eta_j}{\mathcal{F}_j}.\end{aligned}\tag{118}$$

Integrability of this solution implies that

$$\frac{\partial \eta_j}{\partial \tau_j} = -\sqrt{\mathcal{F}_j} \tan \eta_j \frac{\partial \eta_j}{\partial R_j} - \frac{\partial \sqrt{\mathcal{F}_j}}{\partial R_j},\tag{119}$$

but, inserting (118) into the second equation of (82), we find another equation for $\eta(\tau, R, F)$:

$$-\frac{\sin \eta_j}{\sqrt{\mathcal{F}_j}} \frac{\partial \eta_j}{\partial \tau_j} + \cos \eta_j \frac{\partial \eta_j}{\partial R_j} - \sin \eta_j \frac{\partial \ln \mathcal{F}_j}{\partial R_j} + \frac{A_j}{\mathcal{F}_j} \sin \eta_j = 0,\tag{120}$$

which, using the relationship between A and the measure μ derived earlier, simplifies to

$$\frac{\partial \eta_j}{\partial \tau_j} = \sqrt{\mathcal{F}_j} \cot \eta_j \frac{\partial \eta_j}{\partial R_j} + \sqrt{\mathcal{F}_j} \frac{\partial \ln \mu_j}{\partial R_j}.\tag{121}$$

However, equations (121) and (119) are consistent if and only if

$$\tan \eta_j = \frac{\alpha_j}{\mu \sqrt{\mathcal{F}_j}},\tag{122}$$

where $\alpha_j = \alpha_j(\tau_j)$ is a function *only* of τ_j . Taking derivatives with respect to τ_j and R_j respectively and reinserting them into either the integrability condition (119) or into (121) now gives an equation for $\alpha(\tau)$,

$$\frac{\partial \alpha_j}{\partial \tau_j} = \frac{\alpha_j^2}{\mu_j^2} \frac{\partial \mu_j}{\partial R_j} - \frac{\mu_j}{2} \frac{\partial \mathcal{F}_j}{\partial R_j}.\tag{123}$$

Now as α is a function of τ but not of R , and μ, \mathcal{F} are functions of R but not of τ , the above equation requires $\alpha = \text{const.}$ and therefore

$$\frac{\alpha_j^2}{\mu_j^3} \frac{\partial \mu_j}{\partial R_j} = \frac{1}{2} \frac{\partial \mathcal{F}_j}{\partial R_j}.\tag{124}$$

Thus we obtain the solutions (cf. (91) and (92))

$$\mu_j = \frac{\beta_j}{\sqrt{1 - a_j^2 \mathcal{F}_j}}, \quad \mathcal{W}_j = \text{const.} \pm a_j \tau_j \pm \int dR_j \frac{\sqrt{1 - a_j^2 \mathcal{F}_j}}{\mathcal{F}_j},\tag{125}$$

where a_j and $\beta_j = a_j \alpha_j$ are constants. They are unique under the given conditions.

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